Absolute resolvents and masses of irreducible quintic polynomials

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Abstract We describe two new algorithms for determining the Galois group of an irreducible quintic polynomial f defined over a field F. For one approach, we introduce a single degree 24 resolvent polynomial construction, the degrees of whose irreducible factors completely determine the Galois group of f. Our other approach, which does not rely on factoring resolvent polynomials of degree greater than two, considers the discriminant of f along with the size of the automorphism group of the polynomial's stem field. We show that this second method is particularly effective in the case where F is a finite extension of the p-adic numbers.

1 Introduction

Galois theory stands at the intersection of group theory and field theory. In particular, we can consider field extensions obtained by adjoining to a given base field the roots of monic irreducible polynomials. We are often interested in the arithmetic structure of these field extensions, since this information is related to how the roots of these polynomials interact via the algebraic operations of the base field. Galois theory is important because it associates to each polynomial a group (called its Galois group) that encodes this arithmetic structure. For example, one of the most celebrated results of Galois theory states that an irreducible quintic polynomial is solvable by radicals if and only if its Galois group is a subgroup of the metacyclic group $F_{20} \simeq C_5 \rtimes C_4$.

Therefore, an important problem in computational algebra is to determine the Galois group of an irreducible polynomial defined over a field. Algorithms for accomplishing this task have been in existence for more than a century. Indeed, the

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original definition of the Galois group implicitly contained a technique for its determination. For an explicit description of this method, see [18, p.189].

For this paper, we focus on quintic polynomials. Most modern treatments employ a degree 6 resolvent polynomial to determine the Galois group [9, 10]. When this resolvent is factored over the base field, two scenarios can occur. In one case, the resolvent remains irreducible, which indicates the Galois group is either A_5 or S_5, and therefore the original quintic polynomial is not solvable by radicals. In the other case, the degree 6 resolvent factors as a linear times an irreducible quintic, which indicates the Galois group is either cyclic, dihedral, or the previously-mentioned metacyclic group of order 20. To identify the Galois group of the original quintic as a transitive subgroup of S_5, more information is needed. To remedy this, Cohen suggests using two additional resolvent polynomials in [9]; this approach is based on Stauduhar's method of relative resolvents [17].

This paper offers two additional methods for computing Galois groups of quintic polynomials. The first is an improvement of the degree 6 resolvent method. The second incorporates what we call the mass of a polynomial, and it is especially useful over finite extensions of the p-adic numbers.

While our results can be applied to any field of characteristic different from 2, we are mostly interested in applications to number fields and p-adic fields. In Section 2, we improve upon the results found in [9, 10] by situating the degree 6 resolvent method in the larger framework of what has become known as the absolute resolvent method. In this context, we determine all possible absolute resolvent polynomials as well as their factorizations (which depend on the Galois group of the original quintic). In particular, we show that there is a unique absolute resolvent polynomial of degree 24 the degrees of whose irreducible factors completely determine the Galois group of the original quintic polynomial (i.e., no lower degree resolvent polynomial can accomplish this). Lastly, Section 3 introduces a new technique for computing quintic Galois groups based on the notion of the mass of the polynomial. For more examples of the connection between a polynomial's mass and its Galois group see [2, 3, 4, 5, 6, 7]. To provide an example of the versatility of our method, we end the paper by examining Galois groups of totally ramified quintic extensions of p-adic fields for primes $p \neq 2,5$.

2 Absolute Resolvents

Let F be a finite extension of either $\mathbf{Q}$ or $\mathbf{Q}_p$, and let K/F be a quintic extension, defined by the monic irreducible polynomial $f(x)$. Let $\alpha_1,\ldots,\alpha_5$ be the roots of f in some fixed algebraic closure, and let G denote the Galois group of f; i.e., the Galois group of the splitting field of f, or equivalently of the Galois closure of K. Since the elements of G act as permutations on the roots α_i of f, once we fix an ordering of the roots, G can be viewed as a subgroup of S_5 (the symmetric group on 5 letters). Changing the ordering of the roots corresponds to conjugating G in S_5. Since the polynomial f is irreducible, G is a transitive subgroup of S_5; i.e., there is a single orbit for the action of G on the roots α_i of f (each orbit corresponds to an irreducible factor of f).

Therefore, in order to determine the group structure of G, we first identify the conjugacy classes of transitive subgroups of S_5. This information is well known, see [8, 12]).

Since we will make use of all conjugacy classes of subgroups of S_5 (not just the transitive subgroups), we include Table 1 for convenience, which gives information on representatives of the 18 conjugacy classes of nontrivial subgroups of S_5. This information can easily be computed with [12].

Most of the group names in the table are standard. For example, C_n represents the cyclic group of order n, D_n the dihedral group of order $2n$, A_n the alternating group on n letters, and V_4 the Klein 4-group (i.e., C_2^2). There are two different conjugacy classes of C_2, V_4, and S_3 in S_5. In each case, we label one group with an asterisk so as to distinguish the conjugacy classes. Note that the only two groups in this table that are not solvable are A_5 and S_5.

Table 1 Conjugacy classes of nontrivial subgroups of S_5.

Name	Transitive?	Size	Generators
C_2	no	2	(12)
C_2^*	no	2	$(12)(34)$
C_3	no	3	(123)
V_4	no	4	$(12)(34),(13)(24)$
C_4	no	4	$(12)(34),(1324)$
V_4^*	no	4	$(12),(34)$
C_5	yes	5	(12345)
S_3	no	6	$(123),(12)$
S_3^*	no	6	$(123),(12)(45)$
C_3C_2	no	6	$(123),(45)$
D_4	no	8	$(12),(34),(13)(24)$
D_5	yes	10	$(12345),(25)(34)$
A_4	no	12	$(12)(34),(13)(24),(234)$
S_3C_2	no	12	$(123),(23),(45)$
F_{20}	yes	20	$(12345),(25)(34),(2354)$
S_4	no	24	$(12)(34),(13)(24),(234),(34)$
A_5	yes	60	$(12345),(345)$
S_5	yes	120	$(12345),(12)$

2.1 Definition of Resolvents

The usual technique for computing Galois groups involves the notion of absolute resolvent polynomial, which we now define.

Definition 1. Let $T(x_1,\ldots,x_5)$ be a polynomial with integer coefficients. Let H be the stabilizer of T in S_5. That is,

$$H = \left\{\sigma \in S_5 \colon T(x_{\sigma(1)}, \ldots, x_{\sigma(5)}) = T(x_1, \ldots, x_5)\right\}.$$

We define the **resolvent polynomial** $R_{f,T}(x)$ of the polynomial $f(x) \in \mathbf{Z}[x]$ by

$$R_{f,T}(x) = \prod_{\sigma \in S_5/H} \left(x - T(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(5)})\right),$$

where S_5/H is a complete set of right coset representatives of S_5 modulo H and where $\alpha_1, \ldots, \alpha_5$ are the roots of $f(x)$. By Galois theory, $R_{f,T}(x)$ also has integer coefficients.

2.2 Main Theorem on Resolvents

The main theorem concerning resolvent polynomials is the following. A proof can be found in [16].

Theorem 1. *With the notation of the preceding definition, set $m = [S_5 \colon H] = deg(R_{f,T})$. If $R_{f,T}$ is squarefree, its Galois group (as a subgroup of S_m) is equal to $\phi(G)$, where ϕ is the natural group homomorphism from S_5 to S_m given by the natural right action of S_5 on S_5/H.*

Note that we can always ensure $R_{f,T}$ is squarefree by taking a suitable Tschirnhaus transformation of f [9, p.324].

As a consequence, this theorem implies that the list of degrees of irreducible factors of $R_{f,T}$ is the same as the length of the orbits of the action of $\phi(G)$ on the set $[1, \ldots, m]$. In particular, $R_{f,T}$ has a root in F if and only if G is conjugate under S_5 to a subgroup of H.

2.3 Example: Discriminant

Perhaps the most well known example of a resolvent polynomial is the discriminant. Recall that the discriminant of a quintic polynomial $f(x)$ is given by

$$\mathrm{disc}(f) = \prod_{1 \le i < j \le 5} (\alpha_i - \alpha_j)^2,$$

where α_i are the roots of f. In particular, let

$$T = \prod_{1 \le i < j \le 5} (x_i - x_j).$$

It is well-known that T is stabilized by A_5 [11, p. 610]. Notice that a complete set of right coset representatives of S_5/A_5 is $\{(1), (12)\}$. Also notice that applying the permutation (12) to the subscripts of T results in $-T$. In this case, we can form the resolvent polynomial as follows,

$$R_{f,T}(x) = \prod_{\sigma \in S_5/A_5} (x - \sigma(T)) = x^2 - T^2 = x^2 - \mathrm{disc}(f).$$

For example, the discriminant of $f(x) = x^5 + a$ is $5^5 a^4$.

2.4 Example: Degree 6 Resolvent

As another example, consider the standard degree 6 resolvent for quintic polynomials [10]. In this case, we let

$$\begin{aligned} T = & x_1^2 x_2 x_5 + x_1^2 x_3 x_4 + x_2^2 x_1 x_3 + x_2^2 x_4 x_5 + x_3^2 x_1 x_5 \\ & + x_3^2 x_2 x_4 + x_4^2 x_1 x_2 + x_4^2 x_3 x_5 + x_5^2 x_1 x_4 + x_5^2 x_2 x_3. \end{aligned}$$

As proven in [10], the stabilizer of T in S_5 is the group F_{20} of order 20, with generators listed in Table 1. A complete set of right coset representatives for S_5/F_{20} is given by $\{(1),(12),(13),(14),(15),(25)\}$. We can form the degree 6 resolvent for $f(x)$ by computing $R_{f,T}$. For example, if $f(x) = x^5 + 2x + 2$, then the degree 6 resolvent of f is $x^6 + 16x^5 + 160x^4 + 1280x^3 + 6400x^2 - 33616x - 283616$; which we computed using [14] to approximate the roots, then rounding the coefficients of the resolvent.

2.5 Determining the Galois group of a Quintic Polynomial

As stated in Theorem 1, we can use information on how resolvent polynomials factor to determine the Galois group of $f(x)$. Given a transitive subgroup G of S_5, the first step is to determine the factorizations of all possible resolvents arising from a quintic polynomial whose Galois group is G. Theorem 1 shows that this is a purely group-theoretic problem. In particular, the function `resfactors` below will perform such a task. Written for the program GAP [12], the function `resfactors` takes as input a subgroup H of S_5 and a transitive subgroup G of S_5. It computes the permutation representation of G acting on the cosets G/H, and then it outputs the lengths of the orbits of this permutation representation acting on the set $\{1,\ldots,[G:H]\}$. By Theorem 1, the output of the function `resfactors` is precisely the list of degrees of the irreducible factors of $R_{f,T}$ where T is stabilized by H and G is the Galois group of f.

```
resfactors := function(h, g)
  local s5, cosets, index, permrep;
```

```
      s5      := SymmetricGroup(5);
      cosets  := RightCosets(s5,h);
      index   := Size(cosets);
      permrep := Group(List(GeneratorsOfGroup(g),
                   j->Permutation(j, cosets, OnRight)));
    return(List(Orbits(permrep, [1..index]), Size));
  end;
```

Table 2 shows for each conjugacy class of subgroups of S_5 the degrees of the irreducible factors of the corresponding resolvent polynomial according to the Galois group of f.

Table 2 The top row contains the transitive subgroups G of S_5. The left column contains representatives H of conjugacy classes of subgroups of S_5, as in Table 1. For a particular pair (H,G), the entry in the table gives the output of the function `resfactors(H,G)`. In particular, this list is equivalent to the list of degrees of the irreducible factors of the resolvent polynomial $R(T,f)$ where T is stabilized by H and G is the Galois group of the irreducible quintic polynomial f.

H/G	C_5	D_5	F_{20}	A_5	S_5
C_2	5,5,5,5,5,5,5,5,5,5,5,5	10,10,10,10,10,10	20,20,20	60	60
C_2^*	5,5,5,5,5,5,5,5,5,5,5,5	5,5,5,5,10,10,10,10	10,10,20,20	30,30	60
C_3	5,5,5,5,5,5,5,5	10,10,10,10	20,20	20,20	40
V_4	5,5,5,5,5,5	5,5,5,5,5,5	10,10,10	15,15	30
C_4	5,5,5,5,5,5	5,5,10,10	5,5,20	30	30
V_4^*	5,5,5,5,5,5	5,5,10,10	10,20	30	30
C_5	1,1,1,1,5,5,5,5	2,2,10,10	4,20	12,12	24
S_3	5,5,5,5	10,10	20	20	20
S_3^*	5,5,5,5	5,5,5,5	10,10	10,10	20
C_3C_2	5,5,5,5	10,10	20	20	20
D_4	5,5,5	5,5,5	5,10	15	15
D_5	1,1,5,5	1,1,5,5	2,10	6,6	12
A_4	5,5	5,5	10	5,5	10
S_3C_2	5,5	5,5	10	10	10
F_{20}	1,5	1,5	1,5	6	6
S_4	5	5	5	5	5
A_5	1,1	1,1	2	1,1	2

We can use the information in Table 2 to develop algorithms for computing Galois groups of irreducible quintic polynomials. For example, let f be an irreducible quintic polynomial, G the Galois group of f, and g the resolvent for F_{20}. This resolvent is the well-known degree 6 resolvent. We see that:

1. if g factors as a linear times a quintic, then either $G = C_5$, D_5, or F_{20}.
2. if g remains irreducible, then $G = A_5$ or S_5.

The standard procedure for remedying the situation in item (2) is to use the discriminant of f (i.e., the resolvent corresponding to A_5). The discriminant also determines when $G = F_{20}$. To distinguish between C_5 and D_5, we can use the resolvent corresponding to the group C_3C_2, and this is the smallest absolute resolvent that accomplishes that purpose. We point out that Cohen makes use of a different method to distinguish between C_5 and D_5, which is based on Stauduhar's relative resolvent method [17]. See [9] for the details.

As another example, we could use the resolvent corresponding to the group C_2^*, which is a degree 60 polynomial. In this case, all Galois groups are distinguished with this one resolvent. Again letting f denote the quintic polynomial, G its Galois group, and g the degree 60 resolvent corresponding to C_2^*, we have:

1. if g factors as twelve quintics, then $G = C_5$.
2. if g factors as four quintics times four decics, then $G = D_5$.
3. if g factors as two decics times two degree 20 polynomials, then $G = F_{20}$.
4. if g factors as two degree 30 polynomials, then $G = A_5$.
5. if g remains irreducible, then $G = S_5$.

The C_2^* resolvent method mentioned above can be considered an improvement over the degree 6 resolvent method, since it completely determines the Galois group of the polynomial using only one resolvent (and nothing else). However, this method can be improved. If we use the resolvent corresponding to the group C_5, then all Galois groups can still be distinguished with this one resolvent polynomial. However, this only requires factoring a degree 24 resolvent; not a degree 60 polynomial like before. Here is a method to construct this degree 24 resolvent.

A complete set of right coset representatives for S_5/C_5 is:

(1), (45), (34), (132), (125), (35), (23), (135), (152), (12), (1243), (143), (145), (1254), (24), (153), (123), (13), (15), (124), (134), (25), (14), (14)(23).

A form which is stabilized by C_5 is

$$T(x_1,x_2,x_3,x_4,x_5) = x_1x_2^2 + x_2x_3^2 + x_3x_4^2 + x_4x_5^2 + x_5x_1^2.$$

An algorithm for determining quintic Galois groups based on the degree 24 resolvent proceeds as follows. Letting f denote the quintic polynomial, G its Galois group, and g the degree 24 resolvent corresponding to C_5, we have:

1. if g factors as four linears times four quintics, then $G = C_5$.
2. if g factors as two quadratics times two decics, then $G = D_5$.
3. if g factors as one quartic times one degree 20 polynomial, then $G = F_{20}$.
4. if g factors as two dodecics, then $G = A_5$.
5. if g remains irreducible, then $G = S_5$.

For example, consider the Eisenstein polynomial $f(x) = x^5 + 5x + 5$. Forming the degree 24 resolvent polynomial corresponding to the group C_5, we obtain

$$g = x^{24} + 1250x^{21} - 3250x^{20} + \cdots + 2098560333251953125.$$

The resolvent g remains irreducible over $\mathbf{Q}$, indicating the Galois group of f is S_5 in this case. However, g factors as a quartic times a degree 20 polynomial over $\mathbf{Q}_5$ (using [14] for example). Thus the Galois group of f over $\mathbf{Q}_5$ is F_{20}.

3 The Mass of a Polynomial

In the previous section, we situated the standard approach for computing Galois groups of quintic polynomials (the degree 6 resolvent method) into the larger framework of the absolute resolvent method. We completely determined all possible resolvent polynomials along with their factorizations. We then used this information to develop an algorithm to compute Galois groups of quintic polynomials that only relied on factoring a single degree 24 resolvent polynomial.

In this section, we offer a different approach to computing Galois groups of quintic polynomials over p-adic fields. In particular, the aim of this section is to prove the following theorem.

Theorem 2. *Let $p \neq 2,5$ be a prime number and $F/\mathbf{Q}_p$ a finite extension and let f be its residue degree.*

1. *There are five nonisomorphic totally ramified quintic extensions of F with cyclic Galois group if one of the following conditions holds:*
   a. $p \equiv 1 \pmod 5$*;*
   b. $p \equiv -1 \pmod 5$ *and f is even;*
   c. $p \equiv \pm 2 \pmod 5$ *and* $f \equiv 0 \pmod 4$.
2. *There is a unique totally ramified quintic extension of F whose normal closure has Galois group D_5 if one of the following conditions holds:*
   a. $p \equiv -1 \pmod 5$ *and f is odd;*
   b. $p \equiv \pm 2 \pmod 5$ *and* $f \not\equiv 0 \pmod 4$ *and* $\mathbf{Q}_p(\sqrt{5}) \subset F$.
3. *There is a unique totally ramified quintic extension of F whose normal closure has Galois group F_{20} if $p \equiv \pm 2 \pmod 5$ and $f \not\equiv 0 \pmod 4$ and $\mathbf{Q}_p(\sqrt{5}) \not\subset F$.*

Notice that this theorem proves the Galois group of an Eisenstein quintic polynomial defined over the p-adic field F depends only on the prime p, the residue degree of F, and whether or not $\sqrt{5} \in F$. Our approach for proving this theorem is to determine the Galois groups of all possible quintic polynomials of a p-adic field simultaneously, rather than focusing on one polynomial at a time.

Toward that end, we fix a prime $p \neq 2,5$, an algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic numbers, and a finite extension $F/\mathbf{Q}_p$. We let e be the ramification index of F and let f be its residue degree. Thus $ef = [F : \mathbf{Q}_p]$. For a finite extension K/F, we let K^{gal}/F denote its Galois closure, G the Galois group of K^{gal}/F, and $m(K/F)$ the mass of K/F; that is,

$$m(K/F) = [K : F]/|\text{Aut}(K/F)|,$$

where $\text{Aut}(K/F)$ denotes the automorphism group.

3.1 Two Lemmas

In this subsection, we formulate two technical lemmas and describe how they fit together to yield a proof of Theorem 2. First, we focus on the invariants that distinguish between the possible Galois groups for quintic polynomials. One invariant is the discriminant of the polynomial, mentioned previously. If the Galois group G of the polynomial is a subgroup of A_5, then we say the **parity** of G is $+$. As we saw in the previous section, this occurs precisely when the discriminant of the polynomial is a square in F. Otherwise, we say the parity of G is $-$.

Another invariant we use is the centralizer of G in S_5. This quantity is useful for computing Galois groups since it is isomorphic to the automorphism group of the stem field $F[x]/(f(x))$. It turns out that the parity and centralizer order are enough to distinguish between C_5, D_5, and F_{20} (Table 3). Since Galois groups over local fields are solvable, these three Galois groups are the only cases we need to consider [15, Corollary IV.2.5].

Table 3 Parity and centralizer order for the possible Galois groups of quintic extensions of local fields.

G	Parity	$\|C_{S_5}(G)\|$
C_5	$+$	5
D_5	$+$	1
F_5	$-$	1

Our remaining lemmas describe how to compute the mass and centralizer order on the field-theoretic side. The first is a standard result for p-adic fields [13, p. 54].

Lemma 1. *Let $F/\mathbf{Q}_p$ be a finite extension and let n be an integer with $p \nmid n$. Let $g = \gcd(p^f - 1, n)$ and let $m = n/g$.*

(a) There are g nonisomorphic totally ramified extensions of F of degree n; each with mass m.
(b) Let ζ be a primitive $(p^f - 1)$-st root of unity and let π be a uniformizer for F. Each totally and tamely ramified extension of F of degree n is isomorphic to an extension that is generated by a root of the polynomial $x^n + \zeta^r \pi$, for some $0 \le r < g$.

Lemma 2. *Let K/F be a totally ramified extension of degree n with $p \nmid n$ and let $g = \gcd(p^f - 1, n)$. Let $G = Gal(K^{gal}/F)$, where K^{gal} is the normal closure of F. Then*

$$g = |C_{S_n}(G)|.$$

Proof. From Galois theory, we know the automorphism group of K/F is isomorphic to the centralizer of G in S_n. Thus the size of $\mathrm{Aut}(K/F)$ is equal to the order of $C_{S_n}(G)$. Using this fact and the definition of the mass of K/F, we have

$$[K:F] = m(K/F) \cdot |\mathrm{Aut}(K/F)| = m(K/F) \cdot |C_{S_n}(G)|.$$

By Lemma 1, we also have

$$[K:F] = m(K/F)\cdot g.$$

These two equations combine to prove the lemma. □

3.2 Proof of Theorem 2

Proof. We know G must be either C_5, D_5, or F_{20}. Let $g = \gcd(p^f-1,5)$. Thus g is either 1 or 5. Furthermore, $g=5$ if and only if $p^f \equiv 1 \pmod 5$, which occurs if either (a) $p\equiv 1 \pmod 5$, (b) $p\equiv -1 \pmod 5$ and f is even, or (c) $p\equiv \pm 2 \pmod 5$ and $f\equiv 0 \pmod 4$. Since $g = |C_{S_5}(G)|$, we see that $G = C_5$ if and only if one of the conditions (a), (b), or (c) occurs; proving part (1). If $g=1$, then G is either D_5 or F_{20}, depending on whether $\mathrm{disc}(K/F)$ is a square or not, respectively.

Suppose now $g=1$ and that none of (a), (b), and (c) hold. By Lemma 1, the unique totally ramified quintic extension K/F is generated by a root of the polynomial $x^5-\pi$ where π is a uniformizer for F. Since K/F is totally ramified, we have

$$\mathrm{disc}(K/F) = \mathrm{disc}(x^5-\pi) = 5^5\pi^4,$$

which is a square in F if and only if 5 is. Certainly 5 is a square in F if $\mathbf{Q}_p(\sqrt{5})\subset F$.

Suppose $\mathbf{Q}_p(\sqrt{5})\not\subset F$, and consider the polynomial $f(x) = x^2-5$. Since $p\neq 2,5$, Hensel's lemma and quadratic reciprocity show that f has a root in F if and only if $p\equiv \pm 1 \pmod 5$. Since we are supposing that (a) and (b) do not hold, it follows that 5 is a square in F if and only if $p\equiv -1 \pmod 5$ and f is odd or $\mathbf{Q}_p(\sqrt{5})\subset K$. This proves parts (2) and (3). □

We note that with a slight modification to the proof of Theorem 2, the case $p=2$ can be similarly analyzed. When $p=5$, the situation is more complicated, but the details can be extracted from [1].

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