

# CONSTRUCTING GALOIS 2-EXTENSIONS OF THE 2-ADIC NUMBERS

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ABSTRACT. Let  $\mathbf{Q}_2$  denote the field of 2-adic numbers, and let  $G$  be a group of order  $2^n$  for some positive integer  $n$ . We provide an implementation in the software program GAP of an algorithm due to Yamagishi that counts the number of nonisomorphic Galois extensions  $K/\mathbf{Q}_2$  whose Galois group is  $G$ . Furthermore, we describe an algorithm for constructing defining polynomials for each such extension by considering quadratic extensions of Galois 2-adic fields of degree  $2^{n-1}$ . While this method does require that some extensions be discarded, we show that this approach considers far fewer extensions than the best general construction algorithm currently known, which is due to Pauli-Sinclair based on the work of Monge. We end with an application of our approach to completely classify all Galois 2-adic fields of degree 16, including defining polynomials, ramification index, residue degree, valuation of the discriminant, and Galois group.

## 1. INTRODUCTION

Let  $p$  be a prime number and denote by  $\mathbf{Q}_p$  the field of  $p$ -adic numbers. A consequence of Krasner's Lemma is the following: for a fixed positive integer  $n$ , there are only finitely many nonisomorphic extensions of  $\mathbf{Q}_p$  of degree  $n$ . It is therefore natural to ask for a formula which counts extensions; or at least a formula which counts extensions with specified invariants, such as ramification index, residue degree, valuation of the discriminant, Galois group, etc.

Some results are well known. For example, there is a unique unramified extension of  $\mathbf{Q}_p$  of degree  $n$ ; i.e., where the residue degree is  $n$ . Tame ramified extensions are also well understood; that is, extensions where  $p$  does not divide the ramification index. In particular, fix a positive integer  $e$  not divisible by  $p$  and a positive integer  $f$ . Then the number of nonisomorphic tamely ramified extensions of  $\mathbf{Q}_p$  of degree  $n = ef$  with ramification index  $e$  and residue degree  $f$  is given by

$$\sum_{d|f} \phi(f/d) \cdot \gcd(e, p^d - 1),$$

where  $\phi$  is Euler's totient function.

It is also possible to count nonisomorphic extensions of wildly ramified extensions (where  $p$  divides the ramification index) using a formula due to Monge [11]. For example, Table 1 gives the number of nonisomorphic extensions of  $\mathbf{Q}_2$  of degree  $2n$  for  $1 \leq n \leq 16$ . The case  $n = 1$  is well known. The cases  $n = 2, 3, 4, 5$  are classified in [8, 9], where defining polynomials and Galois groups are given. A

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TABLE 1. The number of nonisomorphic extensions of  $\mathbf{Q}_2$  of degree  $2n$  for  $1 \leq n \leq 16$ .

Degree	Number	Degree	Number
2	7	18	2991
4	59	20	314543
6	47	22	8942
8	1823	24	43488949
10	158	26	35294
12	5493	28	19173718
14	590	30	209854
16	890111	32	114799238127

similar classification for the cases  $n = 6, 7$  appears in [1, 3–5]. No other cases have been completely classified.

The first general algorithm for constructing defining polynomials for all nonisomorphic extensions is due to Pauli-Roblot and appears in [14]. Based on recent work by Monge [12], Pauli-Sinclair have developed a faster method for constructing totally ramified extensions of unramified  $p$ -adic fields [13]. In both cases, the algorithms construct a large collection of polynomials some subset of which is guaranteed to define all nonisomorphic extensions. In particular, it is necessary to discard isomorphic extensions. Though we note that the number of polynomials that must be discarded is far smaller in the Pauli-Sinclair method when compared with the Pauli-Roblot method.

If we are interested only in Galois extensions, we can use either of these algorithms. But not only must we discard isomorphic extensions, we also need to discard non-Galois extensions. The purpose of this paper is to offer an alternative approach to constructing defining polynomials of Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  for some positive integer  $n$ . Our method constructs these extensions as quadratic extensions of Galois 2-adic fields of degree  $2^{n-1}$ .

The remainder of the paper is organized as follows. In Section 2 we describe Yamagishi’s algorithm [15] that counts the number of Galois 2-adic fields of degree  $2^n$  with a given 2-group  $G$  as its Galois group. This result is useful for us for two reasons: it allows us to verify when we have found all defining polynomials in our algorithm, and it allows us to demonstrate why our method considers fewer polynomials than the Pauli-Sinclair method (see Table 4 for a comparison). In Section 3 we describe our algorithm for constructing Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  for some positive integer  $n$ . We end with Section 4, where we apply our results to completely classify Galois 2-adic fields of degree 16. We computed defining polynomials for each of the 251 nonisomorphic extensions, the ramification index, residue degree, valuation of the discriminant, and Galois group of each extension. The raw data can be obtained by emailing the first author. We do provide summary tables which count the number of extensions by ramification index, discriminant exponent, and Galois group.

TABLE 2. The number of Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  for  $1 \leq n \leq 6$ .

Degree	Number
2	7
4	19
8	67
16	251
32	915
64	4131

## 2. COUNTING GALOIS 2-EXTENSIONS OF $\mathbf{Q}_2$

In this section, we describe Yamagishi’s algorithm [15] for counting the number of Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  whose Galois group is some specified 2-group  $G$ . We include our implementation of this algorithm for the software program GAP [7]. We then use our algorithm to count Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  for  $1 \leq n \leq 6$ , giving a complete description of the counts for each possible Galois group in the case  $n = 4$ .

As mentioned in [15], the Galois group of the maximal pro-2-extension of  $\mathbf{Q}_2$  is a one-relator group whose relation is given in [10]. Combining this characterization with a well-known enumeration argument, Yamagishi proves the following result. Note, we are restricting Yamagishi’s results to the case where the base field is  $\mathbf{Q}_2$ .

**Theorem 2.1** (Yamagishi). *Let  $G$  be a finite 2-group. The number of nonisomorphic Galois 2-adic fields with Galois group  $G$  is*

$$\frac{1}{\#Aut(G)} \sum_{H \leq G} \mu_G(H) \alpha(H),$$

where  $\#Aut(G)$  denotes the size of the automorphism group of  $G$  and where  $\mu_G$  and  $\alpha$  are defined as follows:

- (1) If  $H \leq G$  with  $[G : H] = 2^i$ , then

$$\mu_G(H) = \begin{cases} (-1)^i 2^{i(i-1)/2} & \text{if } H \geq \Phi(G) \\ 0 & \text{otherwise} \end{cases},$$

where  $\Phi(G)$  denotes the Frattini subgroup of  $G$ .

- (2) For irreducible complex characters  $\chi$  of  $H$ , we have

$$\alpha(H) = \sum_{\chi} \sum_{g, h \in H} \chi(g^2 h^3) \chi(h).$$

Using Theorem 2.1, we can count the number of Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  by running over all possible groups  $G$  of order  $2^n$ . For example, Table 2 contains this data for  $1 \leq n \leq 6$ .

If we focus on the degree 16 Galois extensions of  $\mathbf{Q}_2$ , Table 3 gives more refined information about the 251 extensions by showing how many have a given Galois group. Up to isomorphism, there are 14 groups of order 16. One way to access these groups in GAP is via the TransitiveGroup library. In the table, we identify each group in two ways: (1) by its T-number (accessed by typing `TransitiveGroup(16,T)` in

TABLE 3. For each of the 14 groups of order 16, column **T** gives the group's transitive number, **Name** gives a descriptive name of the group, and **Number** gives the number of nonisomorphic Galois extensions of  $\mathbf{Q}_2$  of degree 16 with the specified group as its Galois group.

<b>T</b>	<b>Name</b>	<b>Number</b>
1	$C_{16}$	48
2	$E_4 \times C_4$	3
3	$E_{16}$	0
4	$C_4 \times C_4$	4
5	$C_8 \times C_2$	36
6	$C_8 \rtimes_5 C_2$	36
7	$Q_8 \times C_2$	3
8	$C_4 \rtimes C_4$	12
9	$D_4 \times C_2$	9
10	$E_4 \rtimes C_4$	12
11	$Q_8 \rtimes C_2$	16
12	$C_8 \rtimes_3 C_2$	36
13	$D_8$	16
14	$Q_{16}$	20

GAP), and (2) by a more descriptive name that indicates its structure. For the descriptive name,  $C_n$  denotes the cyclic group of order  $n$ ,  $E_n$  the elementary abelian group of order  $n$ ,  $D_n$  the dihedral group of order  $2n$ ,  $Q_n$  the generalized quaternion group of order  $n$ ,  $\times$  a direct product, and  $\rtimes$  a semidirect product. In the case of  $C_8 \rtimes C_2$ , there are different mappings from  $C_2$  into  $\text{Aut}(C_8)$ . These give rise to two distinct groups of order 16 (other than  $D_8$ ). One is defined by the mapping  $x \mapsto x^3$ , and the other is given by  $x \mapsto x^5$ . We distinguish these two cases in the obvious way: by  $C_8 \rtimes_3 C_2$  and  $C_8 \rtimes_5 C_2$ , respectively.

We computed the data in Table 3 with the software program GAP [7]. For example, here are several functions we wrote for GAP which will accomplish this task. The main function is `Count2adicFields`, which takes one input, a 2-group  $G$  (called either from GAP's `TransitiveGroup` library or `SmallGroup` library). The other functions are merely auxiliary functions which support the main function, including many for working with character tables to help compute  $\alpha(G)$ .

```
-----
IsEl := function(g,cc)
  return(IsSubset(Elements(cc), [g]));
end;

-----

FindIndex := function(g,cc)
  local j;
  for j in [1..Size(cc)] do
    if
      IsEl(g,cc[j]) then return(j); fi; od;
  return(0);
end;
```

```

end;

-----
EvalChi := function(g,chi,cc)
  local ind;
  ind := FindIndex(g,cc);
  return(chi[ind]);
end;

-----
OneChiSum := function(g,H,chi,cc)
  local lis;
  lis := List(H,j->EvalChi(g^2*j^3,chi,cc)*EvalChi(j,chi,cc));
  return(Sum(lis));
end;

-----
MyChiSum := function(G,H,chi,cc)
  local lis;
  lis := List(G,j->OneChiSum(j,H,chi,cc));
  return(Sum(lis));
end;

-----
MyAlpha := function(g)
  local tbl,cc,c,chi,lis;
  tbl := CharacterTable(g);
  cc := ConjugacyClasses(tbl);
  c := Size(cc);
  chi := Irr(tbl);
  lis := List(chi,j->MyChiSum(g,g,j,cc));
  return(Sum(lis));
end;

-----
MyMu := function(g,h)
  local f,rc,i;
  f := FrattiniSubgroup(g);
  if
    IsSubgroup(h,f) then
      rc := RightCosets(g,h);
      i := LogInt(Size(rc),2);
      return((-1)^i*2^(1/2*i*(i-1)));
    else return(0); fi;
end;

-----
Count2adicFields := function(g)

```

```

local aut,as,lis;
  aut := AutomorphismGroup(g);
  as := AllSubgroups(g);
  lis := List(as,j->MyMu(g,j)*MyAlpha(j));
return(1/Size(aut)*Sum(lis));
end;

```

### 3. OUR ALGORITHM

In this section, we describe our algorithm for producing polynomials defining all nonisomorphic Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$  for some positive integer  $n$ .

Our next result forms the basis of our algorithm for constructing Galois 2-extensions of  $\mathbf{Q}_2$ . In particular, we show that every Galois extension of  $\mathbf{Q}_2$  of degree  $2^n$  can be constructed as a quadratic extension of a Galois extension of  $\mathbf{Q}_2$  of degree  $2^{n-1}$ . This result is an immediate consequence of the following proposition.

**Proposition 3.1.** *Let  $p$  be a prime number and  $K/\mathbf{Q}_p$  a Galois extension of degree  $p^n$  with Galois group  $G$ . There exists a subfield  $F$  of  $K$  with  $[K:F] = p$  such that  $F/\mathbf{Q}_p$  is Galois. Thus  $K$  can be realized as a cyclic degree  $p$  extension of a Galois  $p$ -adic field of degree  $p^{n-1}$ .*

*Proof.* Since  $G$  is a  $p$ -group, its center  $Z(G)$  is nontrivial. Since  $Z(G)$  is also a  $p$ -group, it contains an element of order  $p$  and hence a subgroup  $H$  of order  $p$ . Since  $H \leq Z(G)$ ,  $H$  is a normal subgroup of  $G$ . Let  $F$  be the subfield fixed by  $H$ . Since  $H$  has order  $p$ , the Galois correspondence implies that  $K/F$  is a Galois extension of degree  $p$  with Galois group  $H$ ; hence  $K/F$  is a cyclic degree  $p$  extension. Furthermore, it also follows that  $[F:\mathbf{Q}_p] = p^{n-1}$ . Since  $H$  is normal,  $F/\mathbf{Q}_p$  is Galois; in fact, its Galois group is isomorphic to  $G/H$ .  $\square$

Letting  $p = 2$  in Proposition 3.1, we see that every Galois extension of  $\mathbf{Q}_2$  of degree  $2^n$  can be realized as a quadratic extension of a Galois extension of  $\mathbf{Q}_2$  of degree  $2^{n-1}$ . How many such quadratic extensions are there? The answer is  $2^{2+2^{n-1}} - 1$ .

To see this, let  $K$  be any extension of  $\mathbf{Q}_2$  of degree  $m$ . The quadratic extensions of  $K$  are in one-to-one correspondence with the nontrivial representatives of  $K^*/K^{*2}$ . Let  $O$  denote the ring of integers of  $K$ ,  $P$  its unique prime ideal,  $f$  the residue degree of  $K$ ,  $U$  the units of  $K$  in  $O$  and  $U_1$  the units congruent to 1 modulo  $P$ . Note that  $U_1$  has a natural  $O$ -module structure, being finitely generated of rank  $m$ . Now, we can decompose  $K^*$  in the following way:

$$K^* \simeq \pi^{\mathbf{Z}} \times U \simeq \mathbf{Z} \times \mu_K \times U_1 \simeq \mathbf{Z} \times \mu_K \times \mu_{K,1} \times \mathbf{Z}_2^m,$$

where  $\mu_K$  denotes the  $(2^f - 1)$ -st roots of unity in  $U$ , and  $\mu_{K,1}$  denotes roots of unity congruent to 1 modulo  $P$ . Since  $\mu_K$  has odd order and  $\mu_{K,1}$  has even order, it follows that

$$K^*/K^{*2} \simeq C_2 \times C_2 \times C_2^m \simeq C_2^{2+m}.$$

In Table 4, we compare the number of extensions that our method produces with the number produced by the Pauli-Sinclair algorithm. As the data show, our approach requires much less filtering as the degree of the extensions increase.

TABLE 4. For  $1 \leq n \leq 5$ , Column **Alg** counts the number of extensions produced by Algorithm 3.2 for computing Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$ , **P-S** counts the number of extensions produced by the Pauli-Sinclair method, and **Number** gives the number of nonisomorphic Galois extensions.

Degree	Number	Alg	P-S
2	7	7	7
4	19	105	89
8	67	1,197	4,945
16	251	68,541	4,777,313
32	915	65,797,893	1,221,308,375,461

We point out that quadratic extensions of 2-adic fields are quickly constructed using the algorithm due to Pauli-Roblot [14].

We now describe our algorithm for computing Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$ .

**Algorithm 3.2.** *Let  $n$  be a positive integer, and let  $L$  be the set of Galois extensions of  $\mathbf{Q}_2$  of degree  $2^{n-1}$ . If  $n = 1$ , take  $L = \{\mathbf{Q}_2\}$ . This algorithm produces polynomials defining all nonisomorphic Galois extensions of  $\mathbf{Q}_2$  of degree  $2^n$ .*

- (1) *For each  $K \in L$ , form all quadratic extensions of  $K$  (using [14] for example) and produce polynomials defining each field as an extension over  $\mathbf{Q}_2$ . There should be a total of  $2^{2+2^{n-1}} - 1$  such polynomials.*
- (2) *Discard all extensions that are not Galois. For example, compute the extension's automorphism group and check if the automorphism group has size  $2^n$  (the automorphisms can be represented as roots of linear factors after factoring the polynomial over its stem field).*
- (3) *For each group  $G$  of order  $2^n$ , use Theorem 2.1 to count the number of nonisomorphic extensions of  $\mathbf{Q}_2$  of degree  $2^n$  with Galois group  $G$ .*
- (4) *Partition all extensions by their automorphism groups. For each set in the partition, determine if the number of extensions in the set matches the computation from Step (3). If not, discard isomorphic extensions (using the Root-Finding Algorithm in [14], for example).*
- (5) *Return the list of all remaining polynomials along with their automorphism groups identified either as a transitive subgroup of  $S_{2^n}$  or using the Small-Group libraries located in Magma [6] or GAP.*

In Tables 5–8 we list the outputs of our algorithm for values of  $n$  between 1 and 3. We also include each extension's ramification index, residue degree, and discriminant valuation. Note, we also lowered coefficients of these defining polynomials in an effort to produce polynomials whose coefficients were either 0 or as small as possible.

#### 4. APPLICATION TO GALOIS 2-ADIC FIELDS OF DEGREE 16

The data in Tables 5–8 was already known [8, 9]. In this section, we use Algorithm 3.2 to construct all Galois extensions of  $\mathbf{Q}_2$  of degree 16. To our knowledge, this data has not been computed before; though we do note that one polynomial

TABLE 5. Quadratic extensions of  $\mathbf{Q}_2$ , including a defining polynomial, ramification index  $e$ , residue degree  $f$ , valuation of the discriminant  $c$ , and Galois group  $G$ .

<b>Poly</b>	<b>e</b>	<b>f</b>	<b>c</b>	<b>G</b>
$x^2 + x + 1$	1	2	0	$C_2$
$x^2 + 2x + 2$	2	1	2	$C_2$
$x^2 + 2x + 6$	2	1	2	$C_2$
$x^2 + 2$	2	1	3	$C_2$
$x^2 + 6$	2	1	3	$C_2$
$x^2 + 10$	2	1	3	$C_2$
$x^2 + 14$	2	1	3	$C_2$

TABLE 6. Galois quartic extensions of  $\mathbf{Q}_2$ , including a defining polynomial, ramification index  $e$ , residue degree  $f$ , valuation of the discriminant  $c$ , and Galois group  $G$ .

<b>Poly</b>	<b>e</b>	<b>f</b>	<b>c</b>	<b>G</b>
$x^4 + x + 1$	1	4	0	$C_4$
$x^4 + 2x^3 + 9x^2 + 13$	2	2	4	$E_4$
$x^4 + 8x^3 + 5x^2 + 6x + 1$	2	2	4	$C_4$
$x^4 + 14x^3 + 11x^2 + 14x + 27$	2	2	6	$C_4$
$x^4 + 2x^3 + 5x^2 + 22x + 1$	2	2	6	$C_4$
$x^4 + 18x^3 + 5x^2 + 2x + 17$	2	2	6	$E_4$
$x^4 + 6x^3 + 23x^2 + 10x + 7$	2	2	6	$E_4$
$x^4 + 2x^2 + 4x + 10$	4	1	8	$E_4$
$x^4 + 2x^2 + 4x + 2$	4	1	8	$E_4$
$x^4 + 4x^3 + 2x^2 + 4x + 6$	4	1	8	$E_4$
$x^4 + 4x^3 + 2x^2 + 4x + 14$	4	1	8	$E_4$
$x^4 + 8x^3 + 4x^2 + 10$	4	1	11	$C_4$
$x^4 + 8x^3 + 4x^2 + 2$	4	1	11	$C_4$
$x^4 + 4x^2 + 18$	4	1	11	$C_4$
$x^4 + 4x^2 + 26$	4	1	11	$C_4$
$x^4 + 8x^3 + 4x^2 + 26$	4	1	11	$C_4$
$x^4 + 4x^2 + 2$	4	1	11	$C_4$
$x^4 + 4x^2 + 10$	4	1	11	$C_4$
$x^4 + 8x^3 + 4x^2 + 18$	4	1	11	$C_4$

for each possible Galois group does appear in [2]. The entire set of defining polynomials, along with the extension's ramification index, residue degree, discriminant exponent, and Galois group can be obtained by contacting the first author. Instead, we provide summary tables where we count the number of extensions by ramification index, discriminant exponent, and Galois group. See Tables 9–12.

#### ACKNOWLEDGEMENTS

We thank Brian Sinclair for using his algorithm to verify our degree 16 polynomials in Section 4 defined the same fields as the polynomials produced by the



TABLE 7. Galois octic extensions of  $\mathbf{Q}_2$ , including a defining polynomial, ramification index  $e$ , residue degree  $f$ , valuation of the discriminant  $c$ , and Galois group  $G$ .

Poly	e	f	c	G
$x^8 + x^4 + x^3 + x + 1$	1	8	0	$C_8$
$x^8 + 6x^7 + 11x^6 + 4x^5 + 10x^4 + 14x^3 + 12x^2 + 14x + 1$	2	4	8	$C_8$
$x^8 + 12x^7 + 5x^6 + 8x^5 + 10x^4 + 12x^3 + 8x^2 + 4x + 9$	2	4	8	$C_4 \times C_2$
$x^8 + 8x^7 + 24x^6 + 2x^5 + 14x^4 + 24x^3 + x^2 + 6x + 11$	2	4	12	$C_8$
$x^8 + 12x^7 + 8x^6 + 10x^5 + 22x^4 + 16x^3 + 27x^2 + 14x + 1$	2	4	12	$C_4 \times C_2$
$x^8 + 28x^7 + 16x^6 + 14x^5 + 24x^4 + 24x^3 + 29x^2 + 10x + 17$	2	4	12	$C_4 \times C_2$
$x^8 + 4x^7 + 14x^6 + 18x^5 + 26x^4 + 12x^3 + 21x^2 + 2x + 31$	2	4	12	$C_8$
$x^8 + 20x^7 + 24x^6 + 28x^5 + 17x^4 + 16x^3 + 18x^2 + 20x + 1$	4	2	16	$E_8$
$x^8 + 14x^6 + 8x^5 + 29x^4 + 28x^3 + 16x^2 + 24x + 17$	4	2	16	$C_4 \times C_2$
$x^8 + 16x^7 + 2x^6 + 12x^5 + 5x^4 + 24x^3 + 16x^2 + 20x + 1$	4	2	16	$C_4 \times C_2$
$x^8 + 28x^7 + 31x^4 + 24x^3 + 8x^2 + 28x + 25$	4	2	16	$C_4 \times C_2$
$x^8 + 12x^7 + 28x^6 + 11x^4 + 12x^2 + 20x + 1$	4	2	16	$D_4$
$x^8 + 28x^7 + 12x^6 + 24x^5 + 3x^4 + 24x^3 + 4x^2 + 28x + 1$	4	2	16	$D_4$
$x^8 + 14x^7 + 8x^5 + 15x^4 + 4x^2 + 2x + 5$	4	2	12	$D_4$
$x^8 + 12x^6 + 2x^5 + 15x^4 + 14x^3 + 14x^2 + 2x + 13$	4	2	12	$D_4$
$x^8 + 36x^7 + 62x^6 + 24x^5 + 33x^4 + 40x^3 + 22x^2 + 20x + 51$	4	2	22	$D_4$
$x^8 + 20x^7 + 62x^6 + 24x^5 + 39x^4 + 56x^3 + 50x^2 + 4x + 15$	4	2	22	$C_8$
$x^8 + 12x^7 + 46x^6 + 8x^5 + 41x^4 + 40x^3 + 46x^2 + 44x + 1$	4	2	22	$C_4 \times C_2$
$x^8 + 52x^7 + 50x^6 + 40x^5 + 13x^4 + 40x^3 + 2x^2 + 20x + 33$	4	2	22	$Q_8$
$x^8 + 4x^7 + 46x^6 + 32x^5 + 57x^4 + 16x^3 + 62x^2 + 60x + 59$	4	2	22	$D_4$
$x^8 + 44x^7 + 2x^6 + 48x^5 + 49x^4 + 56x^3 + 6x^2 + 44x + 55$	4	2	22	$C_8$
$x^8 + 12x^7 + 26x^6 + 48x^5 + 9x^4 + 24x^3 + 22x^2 + 60x + 39$	4	2	22	$C_8$
$x^8 + 12x^7 + 22x^6 + 45x^4 + 16x^3 + 14x^2 + 36x + 63$	4	2	22	$C_8$
$x^8 + 20x^7 + 30x^6 + 16x^5 + 15x^4 + 48x^3 + 50x^2 + 44x + 63$	4	2	22	$C_4 \times C_2$
$x^8 + 4x^7 + 2x^6 + 8x^5 + 21x^4 + 32x^3 + 30x^2 + 60x + 35$	4	2	22	$Q_8$
$x^8 + 60x^7 + 10x^6 + 56x^5 + 25x^4 + 40x^3 + 10x^2 + 28x + 17$	4	2	22	$C_4 \times C_2$
$x^8 + 4x^7 + 14x^6 + 8x^5 + 19x^4 + 32x^3 + 46x^2 + 20x + 7$	4	2	22	$C_4 \times C_2$
$x^8 + 8x^5 + 2x^4 + 8x^3 + 4x^2 + 8x + 26$	8	1	24	$D_4$
$x^8 + 8x^7 + 4x^6 + 2x^4 + 8x^3 + 4x^2 + 8x + 30$	8	1	24	$C_4 \times C_2$
$x^8 + 8x^7 + 2x^4 + 4x^2 + 8x + 26$	8	1	24	$C_4 \times C_2$
$x^8 + 8x^7 + 8x^5 + 2x^4 + 8x^3 + 4x^2 + 8x + 18$	8	1	24	$D_4$
$x^8 + 4x^6 + 2x^4 + 4x^2 + 8x + 6$	8	1	24	$Q_8$
$x^8 + 8x^7 + 4x^6 + 2x^4 + 4x^2 + 8x + 30$	8	1	24	$Q_8$
$x^8 + 4x^6 + 2x^4 + 8x^3 + 4x^2 + 8x + 6$	8	1	24	$C_4 \times C_2$

Pauli-Sinclair method. We thank Jonathan Milstead for confirming that our computation of the Galois groups of the degree 16 polynomials matched the result given by his algorithm. We also thank Sebastian Pauli for many helpful discussions.

TABLE 8. Octic extensions of  $\mathbf{Q}_2$  (cont).

Poly	e	f	c	G
$x^8 + 8x^7 + 4x^6 + 2x^4 + 8x^3 + 4x^2 + 8x + 14$	8	1	24	$C_4 \times C_2$
$x^8 + 2x^4 + 4x^2 + 8x + 2$	8	1	24	$C_4 \times C_2$
$x^8 + 4x^6 + 2x^4 + 8x^3 + 4x^2 + 8x + 22$	8	1	24	$C_4 \times C_2$
$x^8 + 4x^6 + 2x^4 + 4x^2 + 8x + 22$	8	1	24	$Q_8$
$x^8 + 2x^4 + 4x^2 + 8x + 18$	8	1	24	$C_4 \times C_2$
$x^8 + 8x^5 + 2x^4 + 8x^3 + 4x^2 + 8x + 10$	8	1	24	$D_4$
$x^8 + 8x^7 + 2x^4 + 4x^2 + 8x + 10$	8	1	24	$C_4 \times C_2$
$x^8 + 8x^7 + 8x^5 + 2x^4 + 8x^3 + 4x^2 + 8x + 2$	8	1	24	$D_4$
$x^8 + 8x^7 + 4x^6 + 2x^4 + 4x^2 + 8x + 14$	8	1	24	$Q_8$
$x^8 + 4x^7 + 10x^4 + 4x^2 + 6$	8	1	22	$D_4$
$x^8 + 4x^7 + 4x^6 + 10x^4 + 4x^2 + 8x + 10$	8	1	22	$D_4$
$x^8 + 4x^7 + 2x^4 + 4x^2 + 14$	8	1	22	$D_4$
$x^8 + 4x^7 + 10x^4 + 4x^2 + 14$	8	1	22	$D_4$
$x^8 + 4x^7 + 4x^6 + 10x^4 + 4x^2 + 8x + 2$	8	1	22	$D_4$
$x^8 + 4x^7 + 4x^6 + 2x^4 + 4x^2 + 8x + 10$	8	1	22	$D_4$
$x^8 + 4x^7 + 4x^6 + 2x^4 + 4x^2 + 8x + 2$	8	1	22	$D_4$
$x^8 + 4x^7 + 2x^4 + 4x^2 + 6$	8	1	22	$D_4$
$x^8 + 8x^6 + 16x^5 + 4x^4 + 50$	8	1	31	$C_8$
$x^8 + 16x^7 + 16x^5 + 4x^4 + 16x^3 + 26$	8	1	31	$C_8$
$x^8 + 16x^5 + 4x^4 + 16x^3 + 10$	8	1	31	$C_8$
$x^8 + 8x^6 + 16x^5 + 4x^4 + 34$	8	1	31	$C_8$
$x^8 + 16x^7 + 8x^6 + 16x^5 + 4x^4 + 34$	8	1	31	$C_8$
$x^8 + 16x^7 + 16x^5 + 4x^4 + 16x^3 + 10$	8	1	31	$C_8$
$x^8 + 8x^6 + 16x^5 + 4x^4 + 18$	8	1	31	$C_8$
$x^8 + 16x^5 + 4x^4 + 16x^3 + 26$	8	1	31	$C_8$
$x^8 + 16x^7 + 8x^6 + 16x^5 + 4x^4 + 2$	8	1	31	$C_8$
$x^8 + 16x^7 + 8x^6 + 16x^5 + 4x^4 + 18$	8	1	31	$C_8$
$x^8 + 8x^6 + 16x^5 + 4x^4 + 2$	8	1	31	$C_8$
$x^8 + 16x^5 + 4x^4 + 16x^3 + 42$	8	1	31	$C_8$
$x^8 + 16x^7 + 16x^5 + 4x^4 + 16x^3 + 42$	8	1	31	$C_8$
$x^8 + 16x^7 + 16x^5 + 4x^4 + 16x^3 + 58$	8	1	31	$C_8$
$x^8 + 16x^5 + 4x^4 + 16x^3 + 58$	8	1	31	$C_8$
$x^8 + 16x^7 + 8x^6 + 16x^5 + 4x^4 + 50$	8	1	31	$C_8$

TABLE 9. The number of Galois extensions of  $\mathbf{Q}_2$  of degree 16 with ramification index 2.

G	c = 16	24	Total
$C_{16}$	1	2	3
$C_8 \times C_2$	1	2	3
<b>Total</b>	2	4	6

TABLE 10. The number of Galois extensions of  $\mathbf{Q}_2$  of degree 16 with ramification index 4.

<b>G</b>	<b>c = 32</b>	<b>36</b>	<b>40</b>	<b>Total</b>
$\mathbf{C}_{16}$			4	4
$\mathbf{E}_4 \times \mathbf{C}_4$		1		1
$\mathbf{C}_4 \times \mathbf{C}_4$			2	2
$\mathbf{C}_8 \times \mathbf{C}_2$		3	2	5
$\mathbf{C}_8 \rtimes_5 \mathbf{C}_2$	1	1	2	4
$\mathbf{C}_4 \rtimes \mathbf{C}_4$			2	2
$\mathbf{E}_4 \rtimes \mathbf{C}_4$	1	1		2
<b>Total</b>	2	6	12	20

TABLE 11. The number of Galois extensions of  $\mathbf{Q}_2$  of degree 16 with ramification index 8.

<b>G</b>	<b>c = 32</b>	<b>36</b>	<b>40</b>	<b>44</b>	<b>48</b>	<b>62</b>	<b>Total</b>
$\mathbf{C}_{16}$						8	8
$\mathbf{E}_4 \times \mathbf{C}_4$					2		2
$\mathbf{C}_4 \times \mathbf{C}_4$					2		2
$\mathbf{C}_8 \times \mathbf{C}_2$					4	8	12
$\mathbf{C}_8 \rtimes_5 \mathbf{C}_2$			4		4	8	16
$\mathbf{Q}_8 \times \mathbf{C}_2$					3		3
$\mathbf{C}_4 \rtimes \mathbf{C}_4$					2		2
$\mathbf{D}_4 \times \mathbf{C}_2$		2		4	3		9
$\mathbf{E}_4 \rtimes \mathbf{C}_4$		2					2
$\mathbf{Q}_8 \rtimes \mathbf{C}_2$			4	4	8		16
$\mathbf{C}_8 \rtimes_3 \mathbf{C}_2$	2	2			4	4	12
$\mathbf{D}_8$					2	2	4
$\mathbf{Q}_{16}$	2	2			2	2	8
<b>Total</b>	4	8	8	8	36	32	96

TABLE 12. The number of Galois extensions of  $\mathbf{Q}_2$  of degree 16 with ramification index 16.

<b>G</b>	<b>c = 54</b>	<b>58</b>	<b>64</b>	<b>79</b>	<b>Total</b>
$\mathbf{C}_{16}$				32	32
$\mathbf{C}_8 \times \mathbf{C}_2$			16		16
$\mathbf{C}_8 \rtimes_5 \mathbf{C}_2$			16		16
$\mathbf{C}_4 \times \mathbf{C}_4$	8				8
$\mathbf{E}_4 \rtimes \mathbf{C}_4$	8				8
$\mathbf{C}_8 \rtimes_3 \mathbf{C}_2$		16	8		24
$\mathbf{D}_8$		8	4		12
$\mathbf{Q}_{16}$		8	4		12
<b>Total</b>	16	32	48	32	128

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